Matrix Structural Analysis

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Abstract

This report will cover the obstacles faced by three undergraduate students, advised by Dr. Nima Rahbar from the University of Massachusetts Dartmouth Civil Department during the summer of 2010. We were sponsored by the Computational Science Training for Undergraduates in the Mathematical Sciences (CSUMS), which was funded by the National Science Foundation (NSF). Our main focus was to study the 2-D Thermal Temperature Distribution throughout a plate specimen produced in Matlab and revised in Octave. Nodes and elements were created which lead to the use of Matrix Structural Analysis. We want to compare our numerical solutions against our exact analytical solutions. I should note that this study is a gateway to more advanced studies to come. We were able to build and manipulate our specimen to fit our advisor's needs.

1 Introduction

People are given many opportunities in life. Depending on what that person does with their opportunity, is what makes that person who they are. Before entering college I was balancing my thoughts based on my previous experience and at the time there was not much experience to balance. Every student before entering college has a choice as to what they should major in. Given the opportunity I was given, I choose Mechanical Engineering. I will now enter my Junior year of college. My first year of college was very interesting. I was granted a special gift without knowledge of this gift. I was to be advised by the soon to be Chairmen of the Mechanical Engineering Department. It was now time to construct my schedule for the following year. I had a chance to let my advisor know that even though my true passion was for Engineering, I also had a desire for numbers, or in this case, mathematics. At the time my advisor, Dr. Peter Friedman, told me that I should pick up a math minor. It took me only seconds to agree and the next thing you know, I was enrolled at the University pursuing a Bachelor of Science Degree in Mechanical Engineering and a Minor in Mathematics.

Shortly after I was introduced to Probability, low and behold, the Chairman of the Math department was the professor. After spending some time with Dr. Ron Tannenwald and discussing my true desire, I soon became a math tutor. Not only was I working with a mass majority of the Math majors, I was also teaching a wide variety of mathematics to a wide variety of students. This was a tremendous feeling. I was able to better my understanding for mathematics and also improve my knowledge in areas I was unfamiliar with. Sadly, the semester was coming to an end. During this time period I was in the process of finding summer work. Gaining the respect and friendship of my co-workers, a couple of good friends of mine now had introduced me to CSUMS.

Well a month later, I was given an opportunity to work with some very intelligent members of the Mathematics Department, given a MacBook and was left to figure the rest out on my own. Finding new ways to learn, communicate, and present my work all came along with the job description. This is where the other two students I was working with and I differed from the rest of the CSUMS community. All three of us were Engineering Students, Vihn Nguyen (Mechanical), Christine Rohr (Civil), and Giuliano Basile (Mechanical). We were to be Advised by a member from the Civil Engineering Department, Dr. Nima Rahbar. Our instructions were simple, Learn Matlab and learn the art of Finite Element Structuring.
2 Finite Element Analysis

Finite Element Analysis (FEA) is a computer animated model used to design and run test creatively and cost effectively. In our case, FEA is used to analyze the specific temperature readings at different nodes created. We are using FEA to meet specifications provided by Dr. Rahbar. FEA can be used to determine the design modifications required to meet our new condition. In general there are two types of analysis used for FEA; 2-D and 3-D modeling. 2-D modeling tends to be less accurate than 3-D modeling but for our purpose we will used 2-D modeling. Our only concern here is the study of temperature distribution throughout our plate specimen.

2.1 How does Finite Element Analysis Work?

FEA uses a system of points called nodes, these nodes form meshes. Structural and material properties have to be programmed into our mesh. Nodes are assigned certain density properties, for reason of simplicity, we will be working with a uniform mesh. Meshes necessarily do not have to be uniform, they can be designed to have higher concentrations of nodes in one part of the mesh then others. Concentrations are used to focus ones studies on a certain part of the specimen they are working with. The mesh acts like a spider web allowing our team to analyze and calculate errors produced between numerical and analytical results.

3 Given Problem

We want to build a plate specimen in Matlab to study the temperature distribution throughout. We will build triangular elements as shown in Figure 1 using Finite Element Structuring. As stated earlier we will use a 2-D model. This specimen is 5 units wide by 10 units high. We are dividing this plate into five equally spaced nodes in both the x and y directions. A heat function is provided and applied to the top of the specimen. The left, right and bottom sides of the plate will be given boundary conditions and will be discussed.

\[
\vartheta = 100 \times \sin \left(\frac{\pi x}{10}\right)
\]  

(1)

![Figure 1: 5x5 Matrix with Linear Triangular Elements](image-url)
3.1 Boundary Conditions

\begin{align*}
\vartheta(x, 0) &= 0 & 0 < x < 5 \\
\vartheta(0, y) &= 0 & 0 < y < 10 \\
\vartheta(x, 10) &= 100 \sin \frac{\pi x}{10} & 0 < x < 5 \\
\frac{\delta \vartheta}{\delta x}(5, y) &= 0 & 0 < y < 10
\end{align*}

3.2 Specimen Details

As seen from lines 2 and 3, we are setting the temperature on the bottom and left sides of the plate equal to zero. We start by taking a look at line 2. With the given parameters of \( \vartheta(x, 0) \), which sets the bottom of the plate equal to zero. This is done by starting at node 0, moving through each node until we reach node 5. We are setting these nodes to zero to represent no initial temperature. The same is done for the parameters in line 3 (left side), except we start from the bottom of the plate and work our way up to the top of the plate.

Moving on to line 4, the Steady State Temperature equation \( \vartheta(x, 10) = 100 \sin \frac{\pi x}{10} \) is being applied to the top nodes of the plate from nodes 0 to 5. Later we will see how this numerical temperature solution compares with the exact analytical temperature solution.

Our final boundary condition can be seen in line 4. \( \frac{\delta \vartheta}{\delta x}(5, y) = 0 \) which says that the flux on the right side of the plate is 0. This means that there is no heat leaving or entering the plate. The only temperature change on the right side of the plate will be due only to the initial function.

4 Application of Research

One may ask, what is the purpose for the study of Steady State Temperature Distribution? There are a multitude of answers. Sometimes Engineers are given design specifications for a plate of some material that has to cover a portion of a high performance engine that gives off a tremendous amount of heat. Common knowledge tells us that millions of dollars are spent to ensure that these engines work properly. Sometimes there are parts in the engine that need to be placed in specific areas due to design specifications. Well let's say that this piece cannot be placed in an environment that exceed a temperature to some degree. Periodically the distance between this piece and let's say the cylinder block is centimeters or even millimeters. This is where the study of Thermal Distribution comes into play. We build our plate specimen in some computer language program, Matlab and study the temperature distribution throughout. After careful analysis we conclude if this plate specimen will be suitable for the design specifications. If not, we conclude that the design has to be reconsidered.

4.1 Material Conditions

Along with design applications comes material specifications. Certain materials have certain conductivities. Conductivity is due to the material's atomic structure. In order to proceed with our study we consider our materials to be at Steady State.

- Metals – Crystalline
- Ceramics – Amorphous
- Polymers – Chains

Atomic structures lead to different Thermal Conductivity or how heat travels throughout a material. Different materials have different temperature distributions. The atomic structures of each material listed above can be seen in Figures 2, 3 and 4. Preceding the figures is a list of some common material's conductivities seen in Table 1. Knowledge of conductivity allows for proper material selection.
Table 1: Thermal Conductivity of common Materials W/(m*K)

<table>
<thead>
<tr>
<th>Materials</th>
<th>Values</th>
</tr>
</thead>
<tbody>
<tr>
<td>Wood</td>
<td>0.04-0.4</td>
</tr>
<tr>
<td>Rubber</td>
<td>0.16</td>
</tr>
<tr>
<td>Polypropylene</td>
<td>0.25</td>
</tr>
<tr>
<td>Cement</td>
<td>0.29</td>
</tr>
<tr>
<td>Glass</td>
<td>1.1</td>
</tr>
<tr>
<td>Soil</td>
<td>1.5</td>
</tr>
<tr>
<td>Steel</td>
<td>12.1-45.0</td>
</tr>
<tr>
<td>Lead</td>
<td>35.3</td>
</tr>
<tr>
<td>Aluminum</td>
<td>237.0</td>
</tr>
<tr>
<td>Gold</td>
<td>318.0</td>
</tr>
<tr>
<td>Silver</td>
<td>429.0</td>
</tr>
<tr>
<td>Diamond</td>
<td>90.0-2320.0</td>
</tr>
</tbody>
</table>

Figure 2: Crystalline Atomic Structure
Figure 3: Chain Atomic Structure
Figure 4: Amorphous Atomic Structure
5 The Process to Building

Let's take another look at Figure 1:

![Figure 5: 5x5 Matrix with Linear Triangular Elements](image)

5.1 Nodes

In order to construct our plate specimen, we need a way to construct our elements contained within our specimen, but in order to construct our elements, we need a way to construct our nodes.

Two different methods are used to position nodes, Local and Global Nodes. Local nodes are used to position global nodes. Referencing Figure 6, let's take a look at element number 1. Local nodes move counter clockwise, start at number 1, and move from the bottom left corner of the triangle to the top of the triangle. Our first plate specimen was built to have 32 elements. In order to construct these elements, 25 Global Nodes were constructed.

Again referencing Figure 6, Element 1, we see Local Node 1, calls Global Node 1. Local node 2 calls Global Node 2. But here's where things become a bit trickier. When calling Local Node 3, we call Global Node 7. This is due to the structure of the triangle. This pattern is seen throughout our specimen.

![Figure 6: Element 1](image)
5.2 Elements

Why use triangle elements? Triangle elements are used to map most 2-D and 3-D objects. Triangles work well when trying to bend around curves, fill irregular shapes, or map a simple rectangular plate specimen.

Since we found a way to create local and global nodes, we now need a way to number our elements. Continuing to focus our attention at Figure 5, 5x5 Matrix with Linear Triangular Elements, We see that Element 1 and Element 2 are created as one complete rectangle. This pattern is followed throughout the row. After each the row has reach its maximum number of elements, the next row is created. This pattern is seen throughout our plate specimen. As a result, we form 32 elements with 25 global nodes. Now that we have an idea of how Nodes and Elements are formulated, lets take a look at how the code relates to our description.

5.3 The Next Node

\[
\text{Xn}=\text{zeros}(\text{nsd, nnp}); \\
\text{pnumb} = 0; \\
\text{for } j = 1: \text{nyd} \\
\quad \text{for } i = 1: \text{nxd} \\
\quad \quad \text{pnumb} = \text{pnumb} + 1; \\
\quad \text{xn}(1, \text{pnumb}) = (i-1) \times \text{xinc}; \\
\quad \text{xn}(2, \text{pnumb}) = (j-1) \times \text{yinc}; \\
\quad \text{end} \\
\text{end}
\]

5.4 The Next Element

\[
\text{ien}=\text{zeros}(\text{nen, nel}); \\
\text{enumb} = 0; \\
\text{for } j = 1: \text{nyd}-1 \\
\quad \text{for } i = 1: \text{nxd}-1 \\
\quad \quad \text{enumb} = \text{enumb} + 1; \\
\quad \quad \text{ien}(1, \text{enumb})= i + (j-1) \times \text{nxd}; \\
\quad \quad \text{ien}(2, \text{enumb})= i + 1 + (j-1) \times \text{nxd}; \\
\quad \quad \text{ien}(3, \text{enumb})= \text{nxd} + i + 1 + (j-1) \times \text{nxd}; \\
\quad \quad \text{enumb} = \text{enumb} + 1; \\
\quad \text{end} \\
\text{end}
\]
6 Numerical vs. Analytical

Now that we have a way to create nodes and elements, its time to study the temperature distribution. Temperature distribution results from the following heat equation.

\[ \vartheta = 100 \sin \frac{\pi x}{10} \]  

(6)

6.1 Numerical Results

As discussed previously in section 3.1 Boundary Conditions, the sine function is applied to the top of our plate specimen. In the following four Figures 7-10, there are four different refined temperature distributions. My teammate, Vinh was able to increased the number of elements within the specimen. As a result, one can see that with the number of elements increasing, the temperature distribution becomes clearer. Later in Section 6.2, Analytical data, we will see how our numerical data compares to our analytical data.

Vinh had decided the new number of element would derive from the the starting number of nodes. As I stated earlier, we created our first plate specimen with a 5x5 Nodal Matrix. In order to produce Figure 8, Vinh added a new node in-between each old node. As a result, nine nodes were calculated and used both in the x and y-direction to create 81 nodes. With the increase number of nodes, we see an increase in the number of elements and as a result we have 128 elements, Figure 8. As the number of elements increase, the temperature distribution become clearer.

![Figure 7: 32 Elements](image7.png)  

![Figure 8: 128 Elements](image8.png)

One might notice the difference between each Figure’s temperature bar. For accuracy, Matlab scales the temperature distribution. This is due to the fact the majority of the temperature change takes place at the top right portion of the plate. For this reason, Matlab scales appropriately. Continuing to focus our attention to Figures 9 and 10, the temperature distribution is clearly seen. Later we will see why 512 Elements will be most suitable.

To create 512 elements 324 nodes were created, 18 nodes in the x and y-direction. To create 1682 elements, 900 nodes were created, 30 nodes in both the x and y-direction.
6.2 Analytical Results

It's time to compare our numerical results produced in Matlab to our Analytical Results produced in Octave. But first let's take some time to answer a few questions...

- What is Numerical data?
- What is Analytical data?
- How do both Numerical and Analytical data compare?

Numerical data is a result of finite element structuring. The resulting elements created allow for an uneven distribution. The reason for this uneven distribution is to represent nature's unpredictable irregularities. Even though we have seen a uniform atomic formation in Figure 2, the Crystalline Atomic Structure, sometimes there are human defects to the material which leads to imperfections. The same can be said for the ceramic's and the polymer's atomic structure.

Analytical data is the result of a function that represents an ideal situation. As we can see in Figures 11-14, no elements are visible because no elements were produced. Each progressing figure corresponds to each progressing figure located in subsection 6.1, Numerical Results. I'd like to give special notice to the increasing clarity of each advancing figure. This is a result of the increase number of nodes. As the plate specimen increases the number of nodal points located inside the plate, the temperature distributions fans out in a smoother pattern.

Now that we have both the Numerical and Analytical results we can make comparisons. Let's start by comparing Figure 7, 32 Elements to Figure 11, 25 Nodes. Both figures contain 25 nodes, but here's where the two figures differ. Figure 7 was made up of 32 elements and Figure 11 was produced without elements. Hence, Figure 11 is considered to be our Analytical result. Both Figures produced data in under ten seconds. Even though the data was really quick, the resolution in Figure 7 and Figure 11 is very low. Later I will introduce errors produced between both figures.

As with Figure 12, 81 Nodes, the data starts to become clearer. But the temperature lines still look as if they are pieced together. This is because the number of nodal points within the specimen are still low. Comparing Figure 12
to Figure 13, we see a dramatic increase in clarity. This is due to the fact that 324 nodes were used (18 nodes in the x and y-direction). The same can be said for Figure 14 where 900 nodes were used (30 nodes in the x and y-direction). Again to reenforce what was said in subsection 6.1, Numerical Results, Figure 12 takes too long to produce, just under eight minutes. This ends up coasting time and money in the long run.

Questions may arise as to why the temperature bars are the same for Figures 11-14? This is due to the fact that the Analytical results were produced in Octave. Unlike Matlab, Octave does not take into consideration the temperature concentration. The distribution of temperature still radiants outwards and still can be considered.
6.3 Computational Time

Increasing the number of elements brings concerns for computational time.

- Is it cost effective to produce such a great number of elements?
- Can optimizations be made?

Let's say a student is taking an engineering class which a project has been assigned. This student has the option to work with three other students. This student knows that if he creates task and distributes them throughout the group, not only will data and results be produced quickly, but also will give time to review data and collaborate. If this student had decided to do the work all on his own, it would take hours, days even weeks.

We can relate this story to computational time needed to complete tasks on the computer using Matlab. More functions needing to be completed, results in longer computation time. If we are able to spread the functions using a method of parallel computing, the results will be faster and more accurate. To answer our first question, it is cost effective to produce a great amount of elements. Figure 10 which shows 1682 Elements, takes too long to compute. Therefore we can say that Figure 9, 512 Elements is suitable. This statement reenforces the comparison of Figures, 13 and 14. Figure 13 uses 324 nodes but when compared to Figure 14 which used 900 nodes, the difference is very small. This difference will be magnified in section 7, Nodal Errors.

6.4 Analytical Code

I have created this code using Octave. The step sizes can be redefined to fit the corresponding number of nodes needed. In this example a 9x9 Nodal Matrix is created and can be related to Figure 12, 81 Nodes.

```matlab
%--------------------------------------------------
% Distribution of Temperature due to the number of nodes
%--------------------------------------------------

steps.x = 9; % Step size in x-direction
steps.y = 9; % Step size in y-direction
x = linspace(0,5,steps.x); % Creates nodes x-axis
tempy = linspace(0,10,steps.y); % Creates nodes y-axis
y = tempy(end:-1:1)'; % Flips array to match diagram layout
u = meshgrid(x); % Creates the mesh
q = meshgrid(y)';

f = @(x,y) (100.*sinh(pi.*y./10).*sin(pi.*x./10))./sinh(pi); % Function used for data
k = f(u,q);
figure(1);
contourf(x,y,k);
axis('equal')
colorbar('west')

7 Nodal Errors

With both numerical and analytical results produced, we can now compute Nodal Error. For example, if we take Figure 7, 32 Elements, which has 25 nodes and compare it to Figure 11, 25 Nodes, we find an error of about 6.2 percent, the greatest of the errors. In Figure 7, each node in the matrix is taken and compared to each node in Figure 11. This
is done by taking the absolute value of the analytical value, subtracted by the numerical value and then dividing this by the analytical value. The following function is done for values in the x and y-direction. A matrix of values is computed and then graphed.

\[ NV - \text{NumericalValue} \]
\[ AV - \text{AnalyticalValue} \]
\[ x = \text{abs}(AV - NV)/AV \]

As we pan over the data produced left to right, we see a decrease in the error readings. This is due to the fact we are increasing the number of elements and nodes. As the number of elements increase the resolution become clearer and the error between the numerical and analytical data becomes less. As discussed previously, here is where producing data for elements greater than 512 is not necessary. The error produced for 512 elements is about .43 percent. The error produced for 1662 elements is about .13 percent. Thats only a 2.3 percent decrease in error. Even though the error produced is smaller for 1682, the difference between the two is not significant enough to see a difference in the data being produced. The time taken to produce the 1683 element figure, is just under 10 minutes, whereas the the time taken to build the 512 element figure is just under 3 minutes.


8 Right Side Temperature Mapping

Referring to section 3.1 Boundary Conditions, one of the conditions was that there was no flux on the right side of the plate. This means that no heat can enter or leave the plate on this side. My teammates and I have decided to map this temperature change along the right side.

Let's start with 32 Elements. In Figure 15 we see a temperature line pieced together with secant lines. These secant lines connect to a few points which represent the nodes on the right side. In this case there are only 5 nodes on the right side so one only see 5 points within this graph.

![Figure 15: Right Side 32 Elements](image)

Here in Figure 16, we see a graph of the right side temperature change for 128 Elements which contains 9 nodes on the right side. This graph is overlaid with Figure 15. The graph begins to smooth out but still nodal points can still be seen in the graph.

![Figure 16: Right Side 128 Elements](image)
In Figure 17, the right side of 512 Element’s temperature change is graphed and placed over Figure 16. There are now 18 nodal points on the right side that can be incorporated in the graph. We see the black temperature line becoming very smooth. This is due to the increase number of nodes on the right side. But this graph is not sufficient enough. This is the only case where we will use more nodal points to make an accurate depiction of what’s really going on.

![Figure 17: Right Side 512 Elements](image)

Finally we see in Figure 18, a nice smooth temperature line. This line is made up of 30 nodal points which allows for such smoothness. This temperature line was taken form the 1682 Element Figure. After analyzing Figures 15-18, we find that the max temperature corresponds to the top right portion of the plate specimen. The minimum temperature begins 0, represents the bottom right side of the plate specimen. We see a nice parabolic temperature decrease from where the initial temperature function was applied. With the increase of nodal points, the smoother the temperature reading becomes. For our purpose of study, 30 nodal points will satisfy.

![Figure 18: Right Side 1682 Elements](image)
9 Future Work

Due to time constraints, our work has been put on hold. There are no limitations with Finite Element Structuring. Most engineering designs if not all designs can be modeled using some computer language. Our work was produced using Matlab and Octave. But FEA is not limited to these computer programs.

I would like to see our work continued. My research team has decided we should make a hole in our specimen and then study the temperature distribution throughout. This plate specimen was produced in its most simplest form, in order to walk you have to crawl. I would like to see how square elements compare to triangle elements. Through research has shown and also through simple geometry, triangle should prove to be superior.

To reenforce our initial question asked in section 4 Application of Research, “What is the purpose for the study of Steady State Temperature Distribution?” With our current study conducted, my teammates and I were able to produce valuable data which can be used for real world applications. Assignments cannot be assigned without following certain parameters. Once parameters have been set, a team of engineers, scientists, mathematicians are able to decipher the problem.

The data provided was produced using Matlab and Octave. Without these mathematical computer programs, non of our work could have been produced. We have proven that triangle elements are best for mapping 2-D models.

10 References

10.1


10.2

"Thermal Conductivity.” Wikipedia, the Free Encyclopedia.